

Ergodic BSDEs driven by G -Brownian motion and their applications

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Abstract

In this paper we consider a new kind of backward stochastic differential equations (BSDEs) driven by G -Brownian motion, called ergodic G -BSDEs. First we establish the uniqueness and existence theorem of G -BSDEs with infinite horizon. Next, we prove the Feynman-Kac formula for fully nonlinear elliptic partial differential equations (PDEs). In particular, we give a new method to prove the uniqueness of viscosity solution to elliptic PDEs. Then we obtain the existence of solutions to G -EBSDEs and the link with fully nonlinear ergodic elliptic PDEs. Finally, we apply these results to the problems of large time behaviour of solutions to fully nonlinear PDEs and optimal ergodic control under model uncertainty.

Key words: G -Brownian motion, ergodic G -BSDEs, elliptic PDEs

MSC-classification: 60H10, 60H30

1 Introduction

In 1990, Pardoux-Peng [18] established the existence and uniqueness theorem for nonlinear BSDEs, which generalize the linear ones of Bismut [3]. Since this pioneering work, the theory of BSDEs has become a useful tool to solve the problems of mathematical finance (see [6, 10]), stochastic control (see [21]) and PDEs (see [17, 19]).

It is well known that BSDEs with a deterministic terminal time provide probabilistic representation for solutions to quasi-linear parabolic PDEs, whereas the BSDEs with a random terminal time are connected with quasi-linear elliptic PDEs (see [4, 20, 27]). The BSDEs with infinite interval can be seen as a special case of BSDEs with a random terminal time. In particular, Fuhrman-Hu-Tessitore [11] (see also [8, 26]) introduced the notion of (markovian) ergodic BSDEs (EBSDEs) through BSDEs with infinite interval (in infinite dimension):

$$Y_s^x = Y_T^x + \int_s^T [f(X_r^x, Z_r^x) - \lambda] dr - \int_s^T Z_r^x dW_r,$$

where $(W_r)_{r \geq 0}$ is a cylindrical Wiener process in a Hilbert space and X^x is the solution to a forward stochastic differential equation starting at x and with values in a Banach space. In this case, λ is the “ergodic cost”. The EBSDEs provide an efficient alternative tool to study optimal control problems with ergodic cost functionals that is functionals depending only on the asymptotic behavior of the state (see also [1, 2]). Moreover, by virtue of a EBSDE approach, Hu-Madec-Richou [15] study the large time behaviour of mild solutions to quasi-linear PDEs (in infinite dimension). We also refer to

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the talk given by Huy  n Pham at the “7th International Symposium on BSDEs” for the study of long time asymptotics of fully nonlinear Bellman equations.

Recently, Peng (2006) introduced the G -expectation theory (see [24, 25] and the references therein). Under the G -expectation framework, the notion of G -Brownian motion and the corresponding stochastic calculus of It  ’s type were also established. In particular, the existence and uniqueness theorem for G -BSDEs and nonlinear Feynman-Kac formula for fully nonlinear PDEs was obtained in [12, 13].

The aim of this paper is to study the following type of (markovian) BSDEs driven by G -Brownian motion with infinite horizon that we shall call G -EBSDEs: for all $0 \leq s \leq T < \infty$,

$$Y_s^x = Y_T^x + \int_s^T [f(X_r^x, Z_r^x) + \gamma^1 \lambda] dr + \int_s^T [g_{ij}(X_r^x, Z_r^x) + \gamma_{ij}^2 \lambda] d\langle B^i, B^j \rangle_r - \int_s^T Z_r^x dB_r - (K_T^x - K_s^x), \quad (1)$$

where γ^1 is a fixed constant and γ^2 is a given $d \times d$ symmetric matrix satisfying $\gamma^1 + 2G(\gamma^2) < 0$, $(B_t)_{t \geq 0}$ is a d -dimensional G -Brownian motion and X^x is the solution to a stochastic differential equation driven by G -Brownian motion starting at x . Our aim is to find a quadruple (Y, Z, K, λ) , where Y, Z are adapted processes, K is a decreasing G -martingale and λ is a real number.

First, we introduce a new kind of linearization method to show that the BSDEs driven by G -Brownian motion with infinite horizon have a unique solution under some certain conditions. However, the linearization methods in [4] and [13] cannot be applied to deal with this problem. In addition, we obtain the comparison theorem for G -BSDEs with infinite horizon. Next, we establish the nonlinear Feynman-Kac formula for elliptic PDEs and introduce a new method to show the uniqueness of viscosity solution to elliptic PDEs in \mathbb{R}^n . Then we show that the G -EBSDEs (1) have a solution (Y^x, Z^x, K^x, λ) . In particular, the G -EBSDEs (1) are connected with the following ergodic elliptic PDEs:

$$G(H(D_x^2 v, D_x v, \lambda, x)) + \langle b(x), D_x v \rangle + f(x, \langle \sigma_1(x), D_x v \rangle, \dots, \langle \sigma_d(x), D_x v \rangle) + \gamma^1 \lambda = 0.$$

This is a fully nonlinear PDE in $(D_x^2 v, \lambda)$, which is a new kind of PDEs in the existence literature. Moreover, this approach provides a tool to study large time behaviour of solutions to fully nonlinear PDE and optimal ergodic control under model uncertainty.

The paper is organized as follows. In section 2, we present some preliminaries for G -BSDEs. The existence and uniqueness theorem for G -BSDEs with infinite horizon is established in section 3. In section 4, we obtain the Feynman-Kac formula for fully nonlinear elliptic PDEs. Section 5 is devoted to study the G -EBSDEs. In section 6, we state some applications of G -EBSDEs.

2 Preliminaries

The main purpose of this section is to recall some basic notions and results of G -expectation, which are needed in the sequel. The readers may refer to [12], [13], [22], [23], [24] for more details.

Let $\Omega = C_0([0, \infty); \mathbb{R}^d)$, the space of \mathbb{R}^d -valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$, be endowed with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} [(\max_{t \in [0, N]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

and B be the canonical process. For each $T > 0$, denote

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\}, \quad L_{ip}(\Omega) := \bigcup_T L_{ip}(\Omega_T),$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of all bounded Lipschitz functions defined on \mathbb{R}^n . For any given monotonic and sublinear function $G : \mathbb{S}_d \rightarrow \mathbb{R}$, let $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}, \hat{\mathbb{E}}_t)$ be the G -expectation space, where $G(A) = \frac{1}{2}\hat{\mathbb{E}}[\langle AB_1, B_1 \rangle]$ and \mathbb{S}_d denotes the space of all $d \times d$ symmetric matrices. In this paper, we shall only consider non-degenerate G -Brownian motion, i.e., there exist some constants $0 < \underline{\sigma}^2 \leq \bar{\sigma}^2 < \infty$ such that, for any $A \geq B$

$$\frac{1}{2}\underline{\sigma}^2 \text{tr}[A - B] \leq G(A) - G(B) \leq \frac{1}{2}\bar{\sigma}^2 \text{tr}[A - B].$$

Denote by $L_G^p(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $(\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $\xi \in L_{ip}(\Omega)$ and $p \geq 1$. Denis et al. [9] proved that the completions of $C_b(\Omega)$ (the set of bounded continuous function on Ω) and $L_{ip}(\Omega)$ under $\|\cdot\|_{L_G^p}$ are the same and we denote them by $L_G^p(\Omega)$. Similarly, we can define $L_G^p(\Omega_T)$ for each $T > 0$.

Theorem 2.1 ([9, 14]) *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega)$, the set of probability measures on $(\Omega, \mathcal{B}(\Omega))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L_G^1(\Omega).$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

Let \mathcal{P} be a weakly compact set that represents $\hat{\mathbb{E}}$. For this \mathcal{P} , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s..

Definition 2.2 *Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \dots, N-1$. For each $p \geq 1$, denote by $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norm $|\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]|^{1/p}$.

For two processes $\eta \in M_G^2(0, T)$ and $\xi \in M_G^1(0, T)$, G -Itô integrals $\int_0^\cdot \eta_s dB_s^i$ and $\int_0^\cdot \xi_s d\langle B^i, B^j \rangle_s$ are well defined, see Li-Peng [16] and Peng [24]. Let $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0, T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$. Denote by $S_G^p(0, T)$ the completion of $S_G^0(0, T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G -BSDEs (in this paper we always use Einstein convention) for a fixed $T > 0$:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (2)$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, \infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

satisfy the following properties:

(H1) There exists a constant $\beta > 0$ such that for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^{2+\beta}(0, n)$ for each $n > 0$;

(H2) There exists a constant $L_1 > 0$ such that

$$|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L_1(|y - y'| + |z - z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^2(0, T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^2(0, T)$, $Z \in M_G^2(0, T; \mathbb{R}^d)$, K is a decreasing G -martingale with $K_0 = 0$ and $K_T \in L_G^2(\Omega_T)$.

Theorem 2.3 ([12]) Assume that $\xi \in L_G^{2+\beta}(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 0$. Then equation (2) has a unique solution $(Y, Z, K) \in \mathfrak{S}_G^2(0, T)$.

We have the following estimates.

Theorem 2.4 ([12]) Let $\xi^l \in L_G^{2+\beta}(\Omega_T)$, $l = 1, 2$, and f^l, g_{ij}^l satisfy (H1) and (H2) for some $\beta > 0$. Assume that $(Y^l, Z^l, K^l) \in \mathfrak{S}_G^2(0, T)$ are the solutions of equation (2) corresponding to ξ^l, f^l and g_{ij}^l . Set $\hat{Y}_t = Y_t^1 - Y_t^2, \hat{Z}_t = Z_t^1 - Z_t^2$. Then there exists a constant C depending on T, G, L_1 such that

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |\hat{Y}_t|^2] &\leq C\{\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^2]] + \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{h}_s ds)^2]]\}, \\ \hat{\mathbb{E}}[\int_0^T |\hat{Z}_s|^2 ds] &\leq C\{\|\hat{Y}\|_{S_G^2}^2 + \|\hat{Y}\|_{S_G^2} \sum_{l=1}^2 [\|Y^l\|_{S_G^2} + \|\int_0^T h_s^{l,0} ds\|_{L_G^2}]\}. \end{aligned}$$

where $\hat{\xi} = \xi^1 - \xi^2$, $\hat{h}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| + \sum_{i,j=1}^d |g_{ij}^1(s, Y_s^2, Z_s^2) - g_{ij}^2(s, Y_s^2, Z_s^2)|$ and $h_s^{l,0} = |f^l(s, 0, 0)| + |g_{ij}^l(s, 0, 0)|$.

We also have the explicit solutions of linear G -BSDEs.

For convenience, assume $d = 1$. Consider the following linear G -BSDE in finite horizon $[0, T]$:

$$Y_t = \xi + \int_t^T f_s ds + \int_t^T g_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3)$$

where $f_s = a_s Y_s + b_s Z_s + m_s, g_s = c_s Y_s + d_s Z_s + n_s$ with $(a_s)_{s \in [0, T]}, (b_s)_{s \in [0, T]}, (c_s)_{s \in [0, T]}, (d_s)_{s \in [0, T]}$ bounded processes in $M_G^2(0, T)$ and $\xi \in L_G^{2+\beta}(\Omega_T)$ for some $\beta > 0, (m_s)_{s \in [0, T]}, (n_s)_{s \in [0, T]} \in M_G^2(0, T)$.

Then we construct an auxiliary extended \tilde{G} -expectation space $(\tilde{\Omega}, L_G^1(\tilde{\Omega}), \hat{\mathbb{E}}^{\tilde{G}})$ with $\tilde{\Omega} = C_0([0, \infty), \mathbb{R}^2)$ and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\sigma^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

Let $(B_t, \tilde{B}_t)_{t \geq 0}$ be the canonical process in the extended space.

Suppose $\{X_t\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t = 1 + \int_0^t a_s X_s ds + \int_0^t c_s X_s d\langle B \rangle_s + \int_0^t d_s X_s dB_s + \int_0^t b_s X_s d\tilde{B}_s. \quad (4)$$

It is easy to verify that

$$X_t = \exp\left(\int_0^t (a_s - b_s d_s) ds + \int_0^t c_s d\langle B \rangle_s\right) \mathcal{E}_t^B \mathcal{E}_t^{\tilde{B}}, \quad (5)$$

where $\mathcal{E}_t^B = \exp(\int_0^t d_s dB_s - \frac{1}{2} \int_0^t d_s^2 d\langle B \rangle_s)$, $\mathcal{E}_t^{\tilde{B}} = \exp(\int_0^t b_s d\tilde{B}_s - \frac{1}{2} \int_0^t b_s^2 d\langle \tilde{B} \rangle_s)$.

Lemma 2.5 ([13]) *In the extended \tilde{G} -expectation space, the solution of the G -BSDE (3) can be represented as*

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s],$$

where $\{X_t\}_{t \in [0, T]}$ is the solution of the \tilde{G} -SDE (4). Moreover,

$$(X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T K_T - \int_t^T a_s K_s X_s ds - \int_t^T c_s K_s X_s d\langle B \rangle_s] = K_t.$$

3 G -BSDEs with infinite horizon

For simplicity, we consider the G -expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ with $\Omega = C_0([0, \infty), \mathbb{R})$ and $\bar{\sigma}^2 = \hat{\mathbb{E}}[B_1^2] \geq -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$. But our results and methods still hold for the case $d > 1$.

This section is devoted to study the following type of BSDEs driven by G -Brownian motion with infinite horizon,

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad 0 \leq t \leq T < \infty, \quad (6)$$

where

$$f(t, \omega, y, z), g(t, \omega, y, z) : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}.$$

In the rest of this section we shall make use of the following assumptions on the generators of G -BSDEs.

(H3) There exists a constant $\mu > 0$ such that $(f(t, \omega, y, z) - f(t, \omega, y', z))(y - y') + 2G((g(t, \omega, y, z) - g(t, \omega, y', z))(y - y')) \leq -\mu|y - y'|^2$.

(H4) $|f(s, 0, 0)| + \bar{\sigma}^2 |g(s, 0, 0)| \leq L_2$ for some constant L_2 .

The following result will be frequently used in this paper.

Lemma 3.1 *For each given $\varepsilon > 0$, there exist four bounded processes $a_s^\varepsilon(y, y', z), b_s^\varepsilon(z, z', y'), c_s^\varepsilon(y, y', z), d_s^\varepsilon(z, z', y')$, such that*

$$\begin{aligned} a_s^\varepsilon(y, y', z) + 2G(c_s^\varepsilon(y, y', z)) &\leq -\mu, \\ |f(s, y, z) - f(s, y', z') - a_s^\varepsilon(y, y', z)(y - y') - b_s^\varepsilon(z, z', y')(z - z')| &\leq 4L_1\varepsilon, \\ |g(s, y, z) - g(s, y', z') - c_s^\varepsilon(y, y', z)(y - y') - d_s^\varepsilon(z, z', y')(z - z')| &\leq 4L_1\varepsilon. \end{aligned}$$

Moreover, for each $T > 0$ and $Y, Y', Z, Z' \in M_G^2(0, T)$, $a_s^\varepsilon(Y_s, Y'_s, Z_s), b_s^\varepsilon(Z_s, Z'_s, Y'_s), c_s^\varepsilon(Y_s, Y'_s, Z_s), d_s^\varepsilon(Z_s, Z'_s, Y'_s) \in M_G^2(0, T)$.

Proof. Denote:

$$\begin{aligned} a_s^\varepsilon(y, y', z) &:= l(y, y', z) \frac{f(s, y, z) - f(s, y', z)}{y - y'} - \frac{\mu}{1 + \underline{\sigma}^2} (1 - l(y, y', z)), \\ c_s^\varepsilon(y, y', z) &:= l(y, y', z) \frac{g(s, y, z) - g(s, y', z)}{y - y'} - \frac{\mu}{1 + \underline{\sigma}^2} (1 - l(y, y', z)), \end{aligned}$$

where $l(y, y', z) = \mathbf{1}_{|y - y'| \geq \varepsilon} + \frac{|y - y'|}{\varepsilon} \mathbf{1}_{|y - y'| < \varepsilon}$. It is obvious that $a_s^\varepsilon(y, y', z), c_s^\varepsilon(y, y', z)$ are continuous functions in (y, y', z) . Thus for each $T > 0$ and $Y, Y', Z \in M_G^2(0, T)$, we can obtain $a_s^\varepsilon(Y_s, Y'_s, Z_s), c_s^\varepsilon(Y_s, Y'_s, Z_s) \in M_G^2(0, T)$.

From assumption (H3), we obtain that

$$\begin{aligned} a_s^\varepsilon(y, y', z) + 2G(c_s^\varepsilon(y, y', z)) &\leq l(y, y', z) \left(\frac{f(s, y, z) - f(s, y', z)}{y - y'} + 2G\left(\frac{g(s, y, z) - g(s, y', z)}{y - y'}\right) \right) \\ &\quad + (1 - l(y, y', z)) \left(-\frac{\mu}{1 + \underline{g}^2} + 2G\left(-\frac{\mu}{1 + \underline{g}^2}\right) \right) \\ &\leq -\mu. \end{aligned}$$

Note that $|a_s^\varepsilon| \leq L_1$ and by assumption (H2), we also derive that

$$\begin{aligned} &|f(s, y, z) - f(s, y', z) - a_s^\varepsilon(y, y', z)(y - y')| \\ &\leq |f(s, y, z) - f(s, y', z)| \mathbf{1}_{|y - y'| < \varepsilon} + |a_s^\varepsilon(y, y', z)(y - y')| \mathbf{1}_{|y - y'| < \varepsilon} \\ &\leq 2L_1|y - y'| \mathbf{1}_{|y - y'| < \varepsilon} \leq 2L_1\varepsilon. \end{aligned}$$

Then set

$$\begin{aligned} b_s^\varepsilon(z, z', y') &:= l(z, z', y') \frac{f(s, y', z) - f(s, y', z')}{z - z'} + L_1(1 - l(z, z', y')), \\ d_s^\varepsilon(z, z', y') &:= l(z, z', y') \frac{g(s, y', z) - g(s, y', z')}{z - z'} + L_1(1 - l(z, z', y')). \end{aligned}$$

One can easily check that the last two inequalities hold. ■

Definition 3.2 A triplet of processes (Y, Z, K) is called a solution of equation (6) if the following properties hold:

- (a) $(Y, Z, K) \in \mathfrak{S}_G^2(0, \infty)$, where $\mathfrak{S}_G^2(0, \infty) = \bigcap_T \mathfrak{S}_G^2(0, T)$;
- (b) $Y_t = Y_T + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$, $0 \leq t \leq T < \infty$.

We now shall state the main result of this section, concerning the existence and uniqueness of solutions of BSDE (6).

Theorem 3.3 Let assumptions (H1)-(H4) hold. Then the G -BSDE (6) has a unique solution (Y, Z, K) belongs to $\mathfrak{S}_G^2(0, \infty)$ such that Y is a bounded process.

Proof. Uniqueness: Suppose that (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) are both solutions of the G -BSDE (6). Set $(\hat{Y}, \hat{Z}) = (Y^1 - Y^2, Z^1 - Z^2)$. Since both Y^1 and Y^2 are bounded continuous processes, we can find some constant $C > 0$ such that $|\hat{Y}| \leq C$. Then we have for any $T > 0$,

$$\hat{Y}_t + K_t^2 = \hat{Y}_T + K_T^2 + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^1 - K_t^1),$$

where $\hat{f}_s = f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)$, $\hat{g}_s = g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)$. From Lemma 3.1, for each given $\varepsilon > 0$, we set $a_s^\varepsilon := a_s^\varepsilon(Y_s^1, Y_s^2, Z_s^1)$. Thus

$$f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^1) = a_s^\varepsilon \hat{Y}_s + f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^1) - a_s^\varepsilon \hat{Y}_s.$$

Moreover, we can get $a_s^\varepsilon \in M_G^2(0, T)$. Similarly, we can define $b_s^\varepsilon, c_s^\varepsilon$ and d_s^ε . Consequently,

$$\hat{f}_s = a_s^\varepsilon \hat{Y}_s + b_s^\varepsilon \hat{Z}_s - m_s^\varepsilon, \quad \hat{g}_s = c_s^\varepsilon \hat{Y}_s + d_s^\varepsilon \hat{Z}_s - n_s^\varepsilon,$$

where $|m_s^\varepsilon| := |\hat{f}_s - a_s^\varepsilon \hat{Y}_s - b_s^\varepsilon \hat{Z}_s| \leq 4L_1\varepsilon$ and $|n_s^\varepsilon| := |\hat{g}_s - c_s^\varepsilon \hat{Y}_s - d_s^\varepsilon \hat{Z}_s| \leq 4L_1\varepsilon$. By Lemma 2.5, we have

$$\begin{aligned} \hat{Y}_t + K_t^2 &= (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon \hat{Y}_T + X_T^\varepsilon K_T^2 - \int_t^T (m_s^\varepsilon + a_s^\varepsilon K_s^2) X_s^\varepsilon ds - \int_t^T (n_s^\varepsilon + c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s] \\ &\leq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon \hat{Y}_T - \int_t^T m_s^\varepsilon X_s^\varepsilon ds - \int_t^T n_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s] + K_t^2, \quad q.s., \end{aligned} \quad (7)$$

where $\{X_t^\varepsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\varepsilon = 1 + \int_0^t a_s^\varepsilon X_s^\varepsilon ds + \int_0^t c_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s + \int_0^t d_s^\varepsilon X_s^\varepsilon dB_s + \int_0^t b_s^\varepsilon X_s^\varepsilon d\tilde{B}_s. \quad (8)$$

Thus from equation (7), we derive that

$$\hat{Y}_t \leq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon \hat{Y}_T] + 4L_1\varepsilon (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [\int_t^T X_s^\varepsilon ds + \int_t^T X_s^\varepsilon d\langle B \rangle_s], \quad q.s.. \quad (9)$$

By equation (5) and Lemma 3.1, we obtain for each $0 \leq t \leq T$,

$$\begin{aligned} (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon] &= \hat{\mathbb{E}}_t^{\tilde{G}} [\exp(\int_t^T (a_s^\varepsilon - b_s^\varepsilon d_s^\varepsilon) ds + \int_t^T c_s^\varepsilon d\langle B \rangle_s) \mathcal{E}_{t,T}^B \mathcal{E}_{t,T}^{\tilde{B}}] \\ &\leq \exp(-\mu(T-t)) \hat{\mathbb{E}}_t^{\tilde{G}} [\exp(-\int_t^T b_s^\varepsilon d_s^\varepsilon ds) \mathcal{E}_{t,T}^B \mathcal{E}_{t,T}^{\tilde{B}}], \quad q.s., \end{aligned}$$

where we have used $\int_t^T c_s^\varepsilon d\langle B \rangle_s - 2 \int_t^T G(c_s^\varepsilon) ds \leq 0$ in the last inequality and $\mathcal{E}_{t,T}^B = \exp(\int_t^T d_s^\varepsilon dB_s - \frac{1}{2} \int_t^T |d_s^\varepsilon|^2 d\langle B \rangle_s)$, $\mathcal{E}_{t,T}^{\tilde{B}} = \exp(\int_t^T b_s^\varepsilon d\tilde{B}_s - \frac{1}{2} \int_t^T |b_s^\varepsilon|^2 d\langle \tilde{B} \rangle_s)$. One can easily check $(\exp(-\int_t^s b_r^\varepsilon d_r^\varepsilon dr) \mathcal{E}_{t,s}^B \mathcal{E}_{t,s}^{\tilde{B}})_{s \geq t}$ is a G -martingale. Then we conclude that

$$(X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon] \leq \exp(-\mu(T-t)), \quad q.s.. \quad (10)$$

By equations (9), (10) and letting $\varepsilon \rightarrow 0$, we deduce that

$$\hat{Y}_t \leq C \exp(-\mu(T-t)), \quad \forall T > 0, \quad q.s..$$

We finally obtain, by sending T to infinity, that $\forall t \geq 0$, $Y_t^1 \leq Y_t^2$, q.s.. By a similar analysis, we also have $Y_t^2 \leq Y_t^1$, q.s.. Thus by the continuity of Y^1 and Y^2 , we conclude that $Y^1 = Y^2$, q.s.. Then from the uniqueness of G -BSDE in finite horizon, we can also get uniqueness of (Z, K) .

Existence: Denote by $(Y^n, Z^n, K^n) \in \mathfrak{S}_G^2(0, n)$ the unique solution of the following G -BSDE in finite horizon:

$$Y_t^n = \int_t^n f(s, Y_s^n, Z_s^n) ds + \int_t^n g(s, Y_s^n, Z_s^n) d\langle B \rangle_s - \int_t^n Z_s^n dB_s - (K_n^n - K_t^n), \quad 0 \leq t \leq n.$$

Using the same method as in the proof of uniqueness, we have

$$Y_t^n = \int_t^n (f(s, 0, 0) + f_s) ds + \int_t^n (g(s, 0, 0) + g_s) d\langle B \rangle_s - \int_t^n Z_s^n dB_s - (K_n^n - K_t^n),$$

where $f_s = f(s, Y_s^n, Z_s^n) - f(s, 0, 0)$, $g_s = g(s, Y_s^n, Z_s^n) - g(s, 0, 0)$. Then for each $\varepsilon > 0$, we can get

$$f_s = a_s^{n,\varepsilon} Y_s^n + b_s^{n,\varepsilon} Z_s^n - m_s^{n,\varepsilon}, \quad g_s = c_s^{n,\varepsilon} Y_s^n + d_s^{n,\varepsilon} Z_s^n - n_s^{n,\varepsilon},$$

where $|m_s^{n,\varepsilon}| \leq 4L_1\varepsilon$ and $|n_s^{n,\varepsilon}| \leq 4L_1\varepsilon$. By Lemma 2.5, we conclude that

$$Y_t^n = (X_t^{n,\varepsilon})^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[\int_t^n (-m_s^\varepsilon + f(s, 0, 0)) X_s^{n,\varepsilon} ds + \int_t^n (-n_s^{n,\varepsilon} + g(s, 0, 0)) X_s^{n,\varepsilon} d\langle B \rangle_s \right], \quad q.s.,$$

where $\{X_t^{n,\varepsilon}\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^{n,\varepsilon} = 1 + \int_0^t a_s^{n,\varepsilon} X_s^{n,\varepsilon} ds + \int_0^t c_s^{n,\varepsilon} X_s^{n,\varepsilon} d\langle B \rangle_s + \int_0^t d_s^{n,\varepsilon} X_s^{n,\varepsilon} dB_s + \int_0^t b_s^{n,\varepsilon} X_s^{n,\varepsilon} d\tilde{B}_s.$$

Thus by equation (10), we derive that

$$\begin{aligned} |Y_t^n| &\leq (X_t^{n,\varepsilon})^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[\int_t^n |f(s, 0, 0)| X_s^{n,\varepsilon} ds + \int_t^n |g(s, 0, 0)| X_s^{n,\varepsilon} d\langle B \rangle_s \right] \\ &\quad + 4L_1\varepsilon (X_t^{n,\varepsilon})^{-1} \hat{E}_t^{\tilde{G}} \left[\int_t^n |X_s^{n,\varepsilon}| ds + \int_t^n |X_s^{n,\varepsilon}| d\langle B \rangle_s \right] \\ &\leq L_2 \int_t^n (X_t^{n,\varepsilon})^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_s^{n,\varepsilon}] ds + 4L_1(1 + \bar{\sigma}^2)\varepsilon \int_t^n (X_t^{n,\varepsilon})^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_s^{n,\varepsilon}] ds \\ &\leq \frac{L_2}{\mu} + 4(1 + \bar{\sigma}^2)\varepsilon \frac{L_1}{\mu}, \quad q.s.. \end{aligned}$$

Then letting $\varepsilon \rightarrow 0$, we can obtain that

$$|Y_t^n| \leq \frac{L_2}{\mu}, \quad q.s.. \quad (11)$$

Now we define Y^n , Z^n and K^n on the whole time axis by setting

$$Y_t^n = Z_t^n = 0, \quad K_t^n = K_n^n, \quad \forall t > n.$$

Fix $t \leq n \leq m$ and set $\tilde{Y} = Y^m - Y^n$, $\tilde{Z} = Z^m - Z^n$. As in the proof of uniqueness, we use the same kind of linearization. Thus

$$\tilde{Y}_t + K_t^m = K_m^m + \int_t^m \tilde{f}_s ds + \int_t^m \tilde{g}_s d\langle B \rangle_s - \int_t^T \tilde{Z}_s dB_s - (K_m^n - K_t^n),$$

where $\tilde{f}_s = f(s, Y_s^m, Z_s^m) - f(s, Y_s^n, Z_s^n) + \mathbf{1}_{s>n} f(s, 0, 0)$, $\tilde{g}_s = g(s, Y_s^m, Z_s^m) - g(s, Y_s^n, Z_s^n) + \mathbf{1}_{s>n} g(s, 0, 0)$. Then for each given $\varepsilon > 0$, we have

$$\tilde{f}_s = a_s^{m,n,\varepsilon} \tilde{Y}_s + b_s^{m,n,\varepsilon} \tilde{Z}_s - m_s^{m,n,\varepsilon} + \mathbf{1}_{s>n} f(s, 0, 0), \quad \hat{g}_s = c_s^{m,n,\varepsilon} \tilde{Y}_s + d_s^{m,n,\varepsilon} \tilde{Z}_s - n_s^{m,n,\varepsilon} + \mathbf{1}_{s>n} g(s, 0, 0),$$

where $|m_s^{m,n,\varepsilon}| \leq 4L_1\varepsilon$ and $|n_s^{m,n,\varepsilon}| \leq 4L_1\varepsilon$. We use the same strategy so as to obtain

$$|\tilde{Y}_t| \leq \frac{L_2}{\mu} \exp(\mu t) (\exp(-\mu n) - \exp(-\mu m)), \quad q.s.. \quad (12)$$

Thus, we get for each $0 < T \leq n \leq m$,

$$\lim_{m,n \rightarrow \infty} \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t^n - Y_t^m|^2 \right] = 0.$$

Consider the following G -BSDE in finite horizon $[0, T]$:

$$Y_t^n = Y_T^n + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T g(s, Y_s^n, Z_s^n) d\langle B \rangle_s - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

By Theorem 2.4, we also conclude that

$$\lim_{m,n \rightarrow \infty} \|Z^n - Z^m\|_{M_G^2(0,T)} = 0.$$

Consequently, there exist two processes $(Y, Z) \in \mathcal{S}_G^2(0, \infty) \times M_G^2(0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt] = 0.$$

Moreover, from equations (11) and (12), we get that $|Y_t| \leq \frac{L_2}{\mu}$ and $|Y_t^n - Y_t| \leq \frac{L_2}{\mu} \exp(-\mu(n - t))$, q.s..

Denote

$$K_t := Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s) ds + \int_0^t g(s, Y_s, Z_s) d\langle B \rangle_s - \int_0^t Z_s dB_s.$$

Then we have $\hat{\mathbb{E}}[|K_t - K_t^n|^2] \rightarrow 0$. We now proceed to prove that K is a G -martingale. For each $0 \leq t < s$, we have

$$\begin{aligned} \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - K_t|] &= \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - \hat{\mathbb{E}}_t[K_s^n] + K_t^n - K_t|] \\ &\leq \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s - K_s^n]|] + \hat{\mathbb{E}}[|K_t^n - K_t|] \\ &= \hat{\mathbb{E}}[|K_s - K_s^n|] + \hat{\mathbb{E}}[|K_t^n - K_t|] \rightarrow 0. \end{aligned}$$

Thus we get $\hat{\mathbb{E}}_t[K_s] = K_t$, which completes the proof. ■

Theorem 3.3 proves the existence and uniqueness of a solution for the BSDE (6). But we also obtained a construction for the solution process (Y, Z, K) . Moreover, we obtain the following comparison theorem of G -BSDEs in infinite horizon.

Theorem 3.4 (Comparison Theorem) *Let (Y^i, Z^i, K^i) be the solution of BSDE (6) with generators f^i and g^i such that Y^i is a bounded process for each $i = 1, 2$. Moreover f^i and g^i satisfy assumptions (H1)-(H4). If $f^1(s, Y_s^i, Z_s^i) - f^2(s, Y_s^i, Z_s^i) + 2G(g^1(s, Y_s^i, Z_s^i) - g^2(s, Y_s^i, Z_s^i)) \leq 0$ for some i , q.s., then for each t , $Y_t^1 \leq Y_t^2$, q.s..*

Proof. Without loss of generality, assume

$$f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2) + 2G(g^1(s, Y_s^2, Z_s^2) - g^2(s, Y_s^2, Z_s^2)) \leq 0.$$

Set $(\hat{Y}, \hat{Z}) = (Y^1 - Y^2, Z^1 - Z^2)$. Since both Y^1 and Y^2 are bounded continuous processes, we can find some constant $C > 0$ such that $|\hat{Y}| \leq C$. Then we have

$$\hat{Y}_t + K_t^2 = \hat{Y}_T + K_T^2 + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^1 - K_t^1),$$

where $\hat{f}_s = f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)$, $\hat{g}_s = g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2)$. By a similar analysis as in the proof of Theorem 3.3, we obtain

$$\hat{f}_s = a_s^\varepsilon \hat{Y}_s + b_s^\varepsilon \hat{Z}_s + m_s - m_s^\varepsilon, \quad \hat{g}_s = c_s^\varepsilon \hat{Y}_s + d_s^\varepsilon \hat{Z}_s + n_s - n_s^\varepsilon,$$

where $|m_s^\varepsilon| \leq 4L_1\varepsilon$, $|n_s^\varepsilon| \leq 4L_1\varepsilon$, $m_s = f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)$ and $n_s = g^1(s, Y_s^2, Z_s^2) - g^2(s, Y_s^2, Z_s^2)$.

Applying Lemma 2.5 yields that

$$\begin{aligned}
\hat{Y}_t + K_t^2 &= (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon (\hat{Y}_T + K_T^2) + \int_t^T (m_s + 2G(n_s) - m_s^\varepsilon - a_s^\varepsilon K_s^2) X_s^\varepsilon ds \\
&\quad + \int_t^T (-n_s^\varepsilon - c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s + \int_t^T n_s X_s^\varepsilon d\langle B \rangle_s - \int_t^T 2G(n_s) X_s^\varepsilon ds] \\
&\leq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon \hat{Y}_T + X_T^\varepsilon K_T^2 - \int_t^T (m_s^\varepsilon + a_s^\varepsilon K_s^2) X_s^\varepsilon ds - \int_t^T (n_s^\varepsilon + c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s] \\
&\leq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} [X_T^\varepsilon \hat{Y}_T - \int_t^T m_s^\varepsilon X_s^\varepsilon ds - \int_t^T n_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s] + K_t^2, \quad q.s.,
\end{aligned}$$

where $\{X_t^\varepsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\varepsilon = 1 + \int_0^t a_s^\varepsilon X_s^\varepsilon ds + \int_0^t c_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s + \int_0^t d_s^\varepsilon X_s^\varepsilon dB_s + \int_0^t b_s^\varepsilon X_s^\varepsilon d\tilde{B}_s.$$

Consequently, we derive that

$$\hat{Y}_t \leq C \exp(-\mu(T-t)), \quad \forall T > 0, \quad q.s..$$

Then sending T to infinity, we have $\forall t \geq 0$, $Y_t^1 \leq Y_t^2$, q.s., which is the desired result. ■

Now we shall establish some stability result. Given a family of functions $(f^\epsilon)_{\epsilon \geq 0}$ and $(g^\epsilon)_{\epsilon \geq 0}$ satisfy (H1)-(H4), with constants L_1, L_2 and μ independent of ϵ . Moreover, for each n

$$\hat{\mathbb{E}}\left[\int_0^n |f^\epsilon(s, Y_s, Z_s) - f^0(s, Y_s, Z_s)|^{2+\beta} ds\right] \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

with some $\beta > 0$.

Theorem 3.5 (Stability Theorem) *Assume a family of functions $(f^\epsilon)_{\epsilon \geq 0}$ and $(g^\epsilon)_{\epsilon \geq 0}$ satisfy the above conditions. Let $(Y^\epsilon, Z^\epsilon, K^\epsilon)$ be the solutions of BSDE (6) with generators f^ϵ and g^ϵ for each ϵ . Then for each $T > 0$,*

$$\lim_{\epsilon \rightarrow 0} \hat{\mathbb{E}}\left[\sup_{s \in [0, T]} |Y_s^\epsilon - Y_s^0|^2 + \int_0^T |Z_s^\epsilon - Z_s^0|^2 ds + |K_T^\epsilon - K_T^0|^2\right] = 0.$$

Proof. Denote by (Y^n, Z^n, K^n) the unique solution of the following G -BSDE in finite horizon:

$$Y_t^{n, \epsilon} = \int_t^n f(s, Y_s^{n, \epsilon}, Z_s^{n, \epsilon}) ds + \int_t^n g(s, Y_s^{n, \epsilon}, Z_s^{n, \epsilon}) d\langle B \rangle_s - \int_t^n Z_s^{n, \epsilon} dB_s - (K_n^{n, \epsilon} - K_t^{n, \epsilon}), \quad 0 \leq t \leq n.$$

By Theorem 3.3, we have for each ϵ

$$|Y_t^\epsilon - Y_t^{n, \epsilon}| \leq \frac{L_2}{\mu} \exp(-\mu(n-t)).$$

Applying Theorem 2.4, we obtain $\|Y^{n, \epsilon} - Y^{n, 0}\|_{\mathcal{S}_{\mathcal{G}}^2(0, T)} \rightarrow 0$ as $\epsilon \rightarrow 0$ for each $T > 0$. Thus we get

$$\lim_{\epsilon \rightarrow 0} \hat{\mathbb{E}}\left[\sup_{s \in [0, T]} |Y_s^\epsilon - Y_s^0|^2\right] = 0,$$

Applying Theorem 2.4 again, we conclude that

$$\lim_{\epsilon \rightarrow 0} \hat{\mathbb{E}}\left[\int_0^T |Z_s^\epsilon - Z_s^0|^2 ds + |K_T^\epsilon - K_T^0|^2\right] = 0,$$

which is the desired result. ■

4 Feynman-Kac formula for fully nonlinear elliptic PDEs

In this section, we give the nonlinear Feynman-Kac Formula for G -BSDEs with infinite horizon. Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function such that $G(A) - G(B) \geq \sigma^2 \text{tr}[A - B]$ for any $A \geq B$ and $B_t = (B_t^i)_{i=1}^d$ be the corresponding d -dimensional G -Brownian motion. Consider the following type of G -FBSDEs with infinite horizon:

$$\begin{cases} X_s^x = x + \int_0^s b(X_r^x) dr + \int_0^s h_{ij}(X_r^x) d\langle B^i, B^j \rangle_r + \int_0^s \sigma(X_r^x) dB_r, \\ Y_s^x = Y_T^x + \int_s^T f(X_r^x, Y_r^x, Z_r^x) dr + \int_s^T g_{ij}(X_r^x, Y_r^x, Z_r^x) d\langle B^i, B^j \rangle_r \\ \quad - \int_s^T Z_r^x dB_r - (K_T^x - K_s^x), \end{cases} \quad (13)$$

where $b, h_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$, $f, g_{ij} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic continuous functions. Consider also the following assumptions:

(B1) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$ for $1 \leq i, j \leq d$, $|f(x, 0, 0)| + 2G(|g_{ij}(x, 0, 0)|)$ is bounded by some constant α ;

(B2) There exist some constants $L, \alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\begin{aligned} |b(x) - b(x')| + \sum_{i,j} |h_{ij}(x) - h_{ij}(x')| &\leq L|x - x'|, \quad |\sigma(x) - \sigma(x')| \leq \alpha_1|x - x'|, \\ |f(x, y, z) - f(x', y', z')| + \sum_{i,j} |g_{ij}(t, x, y, z) - g_{ij}(t, x', y', z')| \\ &\leq L(|x - x'| + |y - y'|) + \alpha_2|z - z'|. \end{aligned}$$

(B3) There exists a constant $\mu > 0$ such that $(f(x, y, z) - f(x, y', z))(y - y') + 2G((g_{ij}(x, y, z) - g_{ij}(x, y', z))(y - y')) \leq -\mu|y - y'|^2$.

(B4) $G(\sum_{i=1}^n (\sigma_i(x) - \sigma_i(x'))^T (\sigma_i(x) - \sigma_i(x')) + 2(\langle x - x', h_{ij}(x) - h_{ij}(x') \rangle)_{i,j=1}^d + \langle x - x', b(x) - b(x') \rangle) \leq -\eta|x - x'|^2$ for some constant $\eta > 0$, where σ_i is the i -th row of σ .

(B5) $\eta - (1 + \bar{\sigma}^2)\alpha_1\alpha_2 > 0$.

The following result is important in our future discussion.

Lemma 4.1 Assume $d = 1$ and suppose \tilde{X} is the solution of the following \tilde{G} -SDE:

$$\tilde{X}_t = 1 + \int_0^t d_s \tilde{X}_s dB_s + \int_0^t b_s \tilde{X}_s d\tilde{B}_s,$$

where $(b_s)_{s \in [0, \infty)}$, $(d_s)_{s \in [0, \infty)}$ are in $M_G^2(0, T)$ for any $T > 0$ and bounded by α_2 . Then the following properties hold:

(i) $\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}| \tilde{X}_t] \leq \exp(-\eta t + (1 + \bar{\sigma}^2)\alpha_1\alpha_2 t)|x - x'|$;

(ii) there exists a constant \bar{C} depending on G, α_1, α_2 and η , such that

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x| \tilde{X}_t] \leq \bar{C}(1 + |x|), \quad \forall t > 0.$$

Proof. It is obvious \tilde{X} is a \tilde{G} -martingale. Then

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}| \tilde{X}_t] \leq \hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}|^2 \tilde{X}_t]^{\frac{1}{2}} \hat{\mathbb{E}}^{\tilde{G}}[\tilde{X}_t]^{\frac{1}{2}} = \hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}|^2 \tilde{X}_t]^{\frac{1}{2}}.$$

Next we shall give the estimate of $|X_t^x - X_t^{x'}|^2 \tilde{X}_t$. Set $C := \eta - (1 + \bar{\sigma}^2)\alpha_1\alpha_2$. Applying the G -Itô formula yields that

$$\begin{aligned} & \exp(2Ct)|X_t^x - X_t^{x'}|^2 \tilde{X}_t - |x - x'|^2 \\ &= 2C \int_0^t \exp(2Cs)|X_s^x - X_s^{x'}|^2 \tilde{X}_s ds + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, b(X_s^x) - b(X_s^{x'}) \rangle \tilde{X}_s ds \\ & \quad + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, h(X_s^x) - h(X_s^{x'}) \rangle \tilde{X}_s d\langle B \rangle_s + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle \tilde{X}_s dB_s \\ & \quad + \int_0^t \exp(2Cs) |\sigma(X_s^x) - \sigma(X_s^{x'})|^2 \tilde{X}_s d\langle B \rangle_s + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 d_s \tilde{X}_s dB_s \\ & \quad + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 b_s \tilde{X}_s d\tilde{B}_s + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle d_s \tilde{X}_s d\langle B \rangle_s \\ & \quad + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle b_s \tilde{X}_s ds \\ &= 2C \int_0^t \exp(2Cs)|X_s^x - X_s^{x'}|^2 \tilde{X}_s ds + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, b(X_s^x) - b(X_s^{x'}) \rangle \tilde{X}_s ds \\ & \quad + 2 \int_0^t \exp(2Cs) G(2\langle X_s^x - X_s^{x'}, h(X_s^x) - h(X_s^{x'}) \rangle + |\sigma(X_s^x) - \sigma(X_s^{x'})|^2) \tilde{X}_s ds \\ & \quad + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle \tilde{X}_s dB_s + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 d_s \tilde{X}_s dB_s \\ & \quad + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 b_s \tilde{X}_s d\tilde{B}_s + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle d_s \tilde{X}_s d\langle B \rangle_s \\ & \quad + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle b_s \tilde{X}_s ds \\ & \quad + \int_0^t \exp(2Cs) (2\langle X_s^x - X_s^{x'}, h(X_s^x) - h(X_s^{x'}) \rangle + |\sigma(X_s^x) - \sigma(X_s^{x'})|^2) \tilde{X}_s d\langle B \rangle_s \\ & \quad - 2 \int_0^t \exp(2Cs) G(2\langle X_s^x - X_s^{x'}, h(X_s^x) - h(X_s^{x'}) \rangle + |\sigma(X_s^x) - \sigma(X_s^{x'})|^2) \tilde{X}_s ds \\ &\leq 2C \int_0^t \exp(2Cs)|X_s^x - X_s^{x'}|^2 \tilde{X}_s ds - 2\eta \int_0^t \exp(2Cs)|X_s^x - X_s^{x'}|^2 \tilde{X}_s ds \\ & \quad + 2 \int_0^t \exp(2Cs) \langle X_s^x - X_s^{x'}, \sigma(X_s^x) - \sigma(X_s^{x'}) \rangle \tilde{X}_s dB_s + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 d_s \tilde{X}_s dB_s \\ & \quad + \int_0^t \exp(2Cs) |X_s^x - X_s^{x'}|^2 b_s \tilde{X}_s d\tilde{B}_s + 2(1 + \bar{\sigma}^2)\alpha_1\alpha_2 \int_0^t \exp(2Cs)|X_s^x - X_s^{x'}|^2 \tilde{X}_s ds, \end{aligned}$$

where we have used $\int_0^t \xi_s d\langle B \rangle_s - 2 \int_0^t G(\xi_s) ds \leq 0$ for any process $\xi_s \in M_G^1(0, T)$ in the last inequality. Then we conclude that

$$\hat{\mathbb{E}}^{\tilde{G}}[\exp(2Ct)|X_t^x - X_t^{x'}|^2 \tilde{X}_t] \leq |x - x'|^2.$$

Consequently,

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}| \tilde{X}_t] \leq \exp(-\eta t + (1 + \bar{\sigma}^2)\alpha_1\alpha_2 t) |x - x'|,$$

and the first inequality holds.

It follows from the G -Itô's formula that

$$\begin{aligned}
& \exp(Ct)|X_t^x|^2 \tilde{X}_t - |x|^2 \\
&= C \int_0^t \exp(Cs)|X_s^x|^2 \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, b(X_s^x) \rangle \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, h(X_s^x) \rangle \tilde{X}_s d\langle B \rangle_s \\
&\quad + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) \rangle \tilde{X}_s dB_s + \int_0^t \exp(Cs) |\sigma(X_s^x)|^2 \tilde{X}_s d\langle B \rangle_s + \int_0^t \exp(Cs) |X_s^x|^2 d_s \tilde{X}_s dB_s \\
&\quad + \int_0^t \exp(Cs) |X_s^x|^2 b_s \tilde{X}_s d\tilde{B}_s + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) \rangle d_s \tilde{X}_s d\langle B \rangle_s + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) \rangle b_s \tilde{X}_s ds \\
&\leq C \int_0^t \exp(Cs) |X_s^x|^2 \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, b(X_s^x) \rangle \tilde{X}_s ds + 2 \int_0^t \exp(Cs) G(2\langle X_s^x, h(X_s^x) \rangle + |\sigma(X_s^x)|^2) \tilde{X}_s ds \\
&\quad + 4 \int_0^t \exp(Cs) G(\langle X_s^x, \sigma(X_s^x) \rangle d_s) \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) \rangle b_s \tilde{X}_s ds + M_t \\
&\leq C \int_0^t \exp(Cs) |X_s^x|^2 \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, b(X_s^x) - b(0) \rangle \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, b(0) \rangle \tilde{X}_s ds \\
&\quad + 2 \int_0^t \exp(Cs) G(|\sigma(X_s^x) - \sigma(0)|^2 + 2\langle X_s^x, h(X_s^x) - h(0) \rangle) \tilde{X}_s ds + M_t \\
&\quad + 4 \int_0^t \exp(Cs) G(\langle \sigma(0), \sigma(X_s^x) - \sigma(0) \rangle + \langle X_s^x, h(0) \rangle) \tilde{X}_s ds + 4 \int_0^t \exp(Cs) G(\langle X_s^x, \sigma(X_s^x) - \sigma(0) \rangle d_s) \tilde{X}_s ds \\
&\quad + 4 \int_0^t \exp(Cs) G(\langle X_s^x, \sigma(0) \rangle d_s) \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) - \sigma(0) \rangle b_s \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(0) \rangle b_s \tilde{X}_s ds \\
&\leq -C \int_0^t \exp(Cs) |X_s^x|^2 \tilde{X}_s ds + 2 \int_0^t \exp(Cs) \langle X_s^x, b(0) \rangle \tilde{X}_s ds + 4 \int_0^t \exp(Cs) G(\langle X_s^x, \sigma(0) \rangle d_s) \tilde{X}_s ds \\
&\quad + 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(0) \rangle b_s \tilde{X}_s ds + 4 \int_0^t \exp(Cs) G(\langle \sigma(0), \sigma(X_s^x) - \sigma(0) \rangle + \langle X_s^x, h(0) \rangle) \tilde{X}_s ds + M_t,
\end{aligned}$$

where $M_t := 2 \int_0^t \exp(Cs) \langle X_s^x, \sigma(X_s^x) \rangle \tilde{X}_s dB_s + \int_0^t \exp(Cs) |X_s^x|^2 d_s \tilde{X}_s dB_s + \int_0^t \exp(Cs) |X_s^x|^2 b_s \tilde{X}_s d\tilde{B}_s$. Note that

$$\begin{aligned}
2\langle X_s^x, b(0) \rangle &\leq \frac{1}{5}C|X_s^x|^2 + 5|b(0)|^2 \frac{1}{C}, \quad 2\langle X_s^x, \sigma(0) \rangle \leq \frac{1}{5\alpha_2}C|X_s^x|^2 + 5\alpha_2|\sigma(0)|^2 \frac{1}{C}, \\
4G(\langle \sigma(0), \sigma(X_s^x) - \sigma(0) \rangle) &\leq \frac{1}{5}C|X_s^x|^2 + 5\alpha_1^2\sigma^4|\sigma(0)|^2 \frac{1}{C}, \quad 4G(\langle X_s^x, h(0) \rangle) \leq \frac{1}{5}C|X_s^x|^2 + 5\bar{\sigma}^4|h(0)|^2 \frac{1}{C}.
\end{aligned}$$

Then taking expectation on both sides, there exists a constant \tilde{C} depending only on $\eta, G, \alpha_1, \alpha_2$ and $b(0), h(0), \sigma(0)$, such that

$$\hat{\mathbb{E}}^{\tilde{G}}[\exp(Ct)|X_t^x|^2 \tilde{X}_t] \leq |x|^2 + \tilde{C} \exp(Ct).$$

Thus, we have

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x| \tilde{X}_t] \leq \sqrt{\tilde{C}} + |x|,$$

which completes the proof. \blacksquare

Under assumptions (B1)-(B5), we define

$$u(x) := Y_0^x, \quad x \in \mathbb{R}^n.$$

Lemma 4.2 *The function u is continuous and bounded. Moreover, there exists some constant M depending only on $L, \eta, \alpha_1, \alpha_2$ and G such that*

$$|u(x) - u(x')| \leq M|x - x'|.$$

Proof. To simplify presentation, we shall prove only the case when $n = d = 1$, as the higher dimensional case can be treated in the same way without substantial difficulty. By Theorem 3.3, Y_t^x is bounded by $\frac{\alpha}{\mu}$. In particular, $|u(x)| \leq \frac{\alpha}{\mu}$. Denote $\tilde{Y} = Y^x - Y^{x'}$, $\tilde{Z} = Z^x - Z^{x'}$. Using the same kind of linearization as in Theorem 3.3, we get

$$\tilde{Y}_t + K_t^{x'} = \tilde{Y}_T + K_T^{x'} + \int_t^T \tilde{f}_s ds + \int_t^T \tilde{g}_s d\langle B \rangle_s - \int_t^T \tilde{Z}_s dB_s - (K_T^x - K_t^x),$$

where $\tilde{f}_s = f(X_s^x, Y_s^x, Z_s^x) - f(X_s^{x'}, Y_s^{x'}, Z_s^{x'})$, $\tilde{g}_s = g(X_s^x, Y_s^x, Z_s^x) - g(X_s^{x'}, Y_s^{x'}, Z_s^{x'})$. Consequently, by Lemma 3.1, we get for each $\epsilon > 0$

$$\tilde{f}_s = a_s^\epsilon \tilde{Y}_s + b_s^\epsilon \tilde{Z}_s + m_s^\epsilon + m_s, \quad \tilde{g}_s = c_s^\epsilon \tilde{Y}_s + d_s^\epsilon \tilde{Z}_s + n_s^\epsilon + n_s,$$

where $m_s = f(X_s^x, Y_s^x, Z_s^x) - f(X_s^{x'}, Y_s^{x'}, Z_s^{x'})$ and $n_s = g(X_s^x, Y_s^x, Z_s^x) - g(X_s^{x'}, Y_s^{x'}, Z_s^{x'})$ and $|m_s^\epsilon| \leq 2(L + \alpha_2)\epsilon$, $|n_s^\epsilon| \leq 2(L + \alpha_2)\epsilon$, $a_s^\epsilon + 2G(c_s^\epsilon) \leq -\mu$. Recalling Lemma 2.5, we obtain that

$$\begin{aligned} \tilde{Y}_0 &\leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^x \tilde{Y}_T + \int_0^T m_s X_s^\epsilon ds + \int_0^T n_s X_s^\epsilon d\langle B \rangle_s] + \hat{\mathbb{E}}^{\tilde{G}}[\int_0^T m_s^\epsilon X_s^\epsilon ds + \int_0^T n_s^\epsilon X_s^\epsilon d\langle B \rangle_s], \\ &\leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^x \tilde{Y}_T + (1 + \bar{\sigma}^2)L \int_0^T |X_s^x - X_s^{x'}| X_s^\epsilon ds] + \hat{\mathbb{E}}^{\tilde{G}}[\int_0^T m_s^\epsilon X_s^\epsilon ds + \int_0^T n_s^\epsilon X_s^\epsilon d\langle B \rangle_s], \quad q.s., \quad (14) \end{aligned}$$

where $\{X_t^\epsilon\}_{t \in [0, T]}$ is given by

$$X_t^\epsilon = \exp(\int_0^t (a_s^\epsilon - b_s^\epsilon d_s^\epsilon) ds + \int_0^t c_s^\epsilon d\langle B \rangle_s) \mathcal{E}_t^B \tilde{\mathcal{E}}_t^{\tilde{B}}.$$

Here $\mathcal{E}_t^B = \exp(\int_0^t d_s^\epsilon dB_s - \frac{1}{2} \int_0^t |d_s^\epsilon|^2 d\langle B \rangle_s)$ and $\tilde{\mathcal{E}}_t^{\tilde{B}} = \exp(\int_0^t b_s^\epsilon d\tilde{B}_s - \frac{1}{2} \int_0^t |b_s^\epsilon|^2 d\langle \tilde{B} \rangle_s)$. Thus

$$|X_t^x - X_t^{x'}| |X_t^\epsilon| \leq \exp(-\mu t) |X_t^x - X_t^{x'}| \tilde{X}_t^\epsilon,$$

where

$$\tilde{X}_t^\epsilon = 1 + \int_0^t d_s^\epsilon \tilde{X}_s^\epsilon dB_s + \int_0^t b_s^\epsilon \tilde{X}_s^\epsilon d\tilde{B}_s.$$

From Lemma 4.1, we conclude that

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_t^x - X_t^{x'}| |X_t^\epsilon|] \leq \exp(-\mu t - \eta t + (1 + \bar{\sigma}^2)\alpha_1 \alpha_2 t) |x - x'|.$$

Thus by equation (14) and sending $\epsilon \rightarrow 0$, we deduce that

$$u(x) - u(x') \leq \exp(-\mu T) \frac{\alpha}{\mu} + \frac{(1 + \bar{\sigma}^2)L}{\mu + \eta - (1 + \bar{\sigma}^2)\alpha_1 \alpha_2} |x - x'|,$$

Letting $T \rightarrow \infty$, we obtain $u(x) - u(x') \leq \frac{(1 + \bar{\sigma}^2)L}{\eta - (1 + \bar{\sigma}^2)\alpha_1 \alpha_2} |x - x'|$. In a similar way, we also have $u(x') - u(x) \leq \frac{(1 + \bar{\sigma}^2)L}{\eta - (1 + \bar{\sigma}^2)\alpha_1 \alpha_2} |x - x'|$, which is the desired result. ■

Now we shall present the main results of this section.

Lemma 4.3 For each t , we have $Y_t^x = u(X_t^x)$.

The proof will be given in the appendix.

Theorem 4.4 $u(x)$ is the unique bounded continuous viscosity solution of the following PDE:

$$G(H(D_x^2 u, D_x u, u, x)) + \langle b(x), D_x u \rangle + f(x, u, \langle \sigma_1(x), D_x u \rangle, \dots, \langle \sigma_d(x), D_x u \rangle) = 0, \quad (15)$$

where

$$H_{ij}(D_x^2 u, D_x u, u, x) = \langle D_x^2 u \sigma_i(x), \sigma_j(x) \rangle + 2\langle D_x u, h_{ij}(x) \rangle + 2g_{ij}(x, u, \langle \sigma_1(x), D_x u \rangle, \dots, \langle \sigma_d(x), D_x u \rangle).$$

Proof. The uniqueness of viscosity solution of equation (15) will be given in appendix. Applying Lemma 4.3, we obtain for each $\delta > 0$,

$$\begin{aligned} u(x) = & u(X_\delta^x) + \int_0^\delta f(X_r^x, Y_r^x, Z_r^x) dr + \int_0^\delta g_{ij}(X_r^x, Y_r^x, Z_r^x) d\langle B^i, B^j \rangle_r \\ & - \int_0^\delta Z_r^x dB_r - K_\delta^x. \end{aligned}$$

Then we can prove that u is a viscosity solution of equation (15) in the same way as in [13]. ■

In the next theorem, we shall discuss the sign of the solution of equation (15).

Theorem 4.5 Suppose moreover that $-f(X_s^x, 0, 0) + 2G(-g_{ij}(X_s^x, 0, 0)) \leq 0$ for each $s > 0$. Then $u(x) \geq 0$.

Proof. It follows from Comparison Theorem 3.4 that $\forall t \geq 0, Y_t^x \geq 0$. In particular, for $t = 0$, we deduce that $u(x) \geq 0$. ■

Remark 4.6 We remark that our results still hold when the coefficients f and g_{ij} are of polynomial growth in x . The proof is the same without any difficulty just changing the condition (B4). For simplicity, we focus on the case that the coefficients satisfy bounded conditions.

5 Ergodic BSDEs driven by G -Brownian motion

In this section, we shall study the following type of (markovian) ergodic BSDEs driven by G -Brownian motion under assumptions (B1), (B2), (B4) and (B5) ($\mu = 0$):

$$Y_s^x = Y_T^x + \int_s^T [f(X_r^x, Z_r^x) + \gamma^1 \lambda] dr + \int_s^T [g_{ij}(X_r^x, Z_r^x) + \gamma_{ij}^2 \lambda] d\langle B^i, B^j \rangle_r - \int_s^T Z_r^x dB_r - (K_T^x - K_s^x), \quad (16)$$

where γ^1 is a fixed constant and γ^2 is a given $d \times d$ symmetric matrix satisfied $\gamma^1 + 2G(\gamma^2) < 0$.

As in [4], we start by considering an infinite horizon equation with strictly monotonic drift, namely for each $\epsilon > 0$, the G -BSDEs:

$$\begin{aligned} Y_s^{x,\epsilon} = & Y_T^{x,\epsilon} + \int_s^T [f(X_r^x, Z_r^{x,\epsilon}) + \gamma^1 \epsilon Y_r^{x,\epsilon}] dr + \int_s^T [g_{ij}(X_r^x, Z_r^{x,\epsilon}) + \gamma_{ij}^2 \epsilon Y_r^{x,\epsilon}] d\langle B^i, B^j \rangle_r \\ & - \int_s^T Z_r^{x,\epsilon} dB_r - (K_T^{x,\epsilon} - K_s^{x,\epsilon}). \end{aligned} \quad (17)$$

From Theorem 3.3, we immediately have

Lemma 5.1 *The G-BSDE (17) has a unique solution $(Y^{x,\epsilon}, Z^{x,\epsilon}, K^{x,\epsilon})$ belongs to $\mathfrak{S}_G^2(0, \infty)$ such that $Y^{x,\epsilon}$ is a bounded process. Furthermore, $|Y_t^{\epsilon,x}| \leq \frac{\alpha}{-(\gamma^1 + 2G(\gamma^2))\epsilon}$.*

Then denote $v^\epsilon(x) := Y_0^{x,\epsilon}$. Then by Lemma 4.2, we have

Lemma 5.2 *There exists some constant $M > 0$ independent of ϵ such that*

$$|v^\epsilon(x) - v^\epsilon(x')| \leq M|x - x'|.$$

Denote $\bar{v}^\epsilon(x) = v^\epsilon(x) - v^\epsilon(0)$. Then $|\bar{v}^\epsilon(x)| \leq M|x|$ and $\epsilon v^\epsilon(0) \leq \frac{\alpha}{-\gamma^1 - 2G(\gamma^2)}$. Note that $\bar{v}^\epsilon(x)$ is a M -Lipschitz function for each ϵ . Thus we can construct by a diagonal procedure a sequence $\epsilon_n \downarrow 0$ such that $\bar{v}^{\epsilon_n}(x) \rightarrow v(x)$ for all $x \in \mathbb{R}^n$ and $\epsilon_n v^{\epsilon_n}(0) \rightarrow \lambda$. It follows from Theorem 4.4 that $\bar{v}^\epsilon(x)$ is the unique viscosity solution of the following PDE:

$$\begin{aligned} G(H(D_x^2 \bar{v}^\epsilon, D_x \bar{v}^\epsilon, \bar{v}^\epsilon + v^\epsilon(0), x)) + \langle b(x), D_x \bar{v}^\epsilon \rangle + f(x, \langle \sigma_1(x), D_x \bar{v}^\epsilon \rangle, \dots, \langle \sigma_d(x), D_x \bar{v}^\epsilon \rangle) \\ = -\gamma^1 \epsilon \bar{v}^\epsilon - \gamma^1 \epsilon v^\epsilon(0), \end{aligned} \quad (18)$$

where

$$\begin{aligned} H_{ij}(D_x^2 \bar{v}^\epsilon, D_x \bar{v}^\epsilon, \bar{v}^\epsilon + v^\epsilon(0), x) = & \langle D_x^2 \bar{v}^\epsilon \sigma_i(x), \sigma_j(x) \rangle + 2 \langle D_x \bar{v}^\epsilon, h_{ij}(x) \rangle \\ & + 2g_{ij}(x, \langle \sigma_1(x), D_x \bar{v}^\epsilon \rangle, \dots, \langle \sigma_d(x), D_x \bar{v}^\epsilon \rangle) + 2\gamma_{ij}^2 \epsilon \bar{v}^\epsilon + 2\gamma_{ij}^2 \epsilon v^\epsilon(0). \end{aligned}$$

Applying the stability result of viscosity solutions (see Proposition 4.3 in [7]), we obtain (v, λ) is a viscosity pair solution of the following fully nonlinear ergodic elliptic PDE:

$$G(H(D_x^2 v, D_x v, \lambda, x)) + \langle b(x), D_x v \rangle + f(x, \langle \sigma_1(x), D_x v \rangle, \dots, \langle \sigma_d(x), D_x v \rangle) + \gamma^1 \lambda = 0, \quad (19)$$

where

$$\begin{aligned} H_{ij}(D_x^2 v, D_x v, \lambda, x) = & \langle D_x^2 v \sigma_i(x), \sigma_j(x) \rangle + 2 \langle D_x v, h_{ij}(x) \rangle \\ & + 2g_{ij}(x, \langle \sigma_1(x), D_x v \rangle, \dots, \langle \sigma_d(x), D_x v \rangle) + 2\gamma_{ij}^2 \lambda. \end{aligned}$$

In addition, v is a M -Lipschitz function.

Remark 5.3 Note that the equation 19 is a fully nonlinear elliptic PDE in $(D_x^2 v, \lambda)$, which is different from the previous works (see [2, 15] and the references therein).

In particular, we have the following probabilistic representation for the fully nonlinear ergodic elliptic PDE (19).

Theorem 5.4 *Suppose assumptions (B1), (B2), (B4) and (B5) hold. Then for each x , the G-EBSDE (16) has a solution $(Y^x, Z^x, K^x, \lambda) \in \mathfrak{S}_G^2(0, \infty) \times \mathbb{R}$ such that $|Y_s^x| \leq M|X_s^x|$.*

Proof. For each x , denote $Y_s^x := v(X_s^x)$. For each $T > 0$, consider the following parabolic PDE:

$$\begin{cases} \partial_t \phi(t, x) + G(H(D_x^2 \phi, D_x \phi, \lambda, x)) + \langle b(x), D_x \phi \rangle + f(x, \langle \sigma_1(x), D_x \phi \rangle, \dots, \langle \sigma_d(x), D_x \phi \rangle) + \gamma^1 \lambda = 0, \\ \phi(T, x) = v(x). \end{cases}$$

From the uniqueness of viscosity solution, we get that $\phi(t, x) = v(x)$. By the nonlinear Feynman-Kac formula in [13], we obtain for each $T > 0$,

$$\begin{aligned} v(X_s^x) = v(X_T^x) + \int_s^T [f(X_r^x, Z_r^{x,T}) + \gamma^1 \lambda] dr + \int_s^T [g_{ij}(X_r^x, Z_r^{x,T}) + \gamma_{ij}^2 \lambda] d\langle B^i, B^j \rangle_r \\ - \int_s^T Z_r^{x,T} dB_r - (K_T^{x,T} - K_s^{x,T}). \end{aligned} \quad (20)$$

By the uniqueness of solution to G -BSDE, it is obvious $(Z_r^{x,T}, K_r^{x,T}) = (Z_r^{x,S}, K_r^{x,S})$ for $S > T$. Set $(Z_t^x, K_t^x) = (Z_t^{x,T}, K_t^{x,T})$ for some $T \geq t$. Then (Y^x, Z^x, K^x, λ) satisfies equation (16). ■

We remark that the solution to G -EBSDE (16) is not unique. Indeed the equation is invariant with respect to addition of a constant to Y . However we have a uniqueness result for λ under some additional condition.

Theorem 5.5 *If for some $x \in \mathbb{R}^n$, $(Y'^x, Z'^x, K'^x, \lambda') \in \mathfrak{S}_G^2(0, \infty) \times \mathbb{R}$ verifies equation (16). Moreover, there exists some constant $c^x > 0$ such that*

$$|Y_s'^x| \leq c^x(1 + |X_s^x|).$$

Then $\lambda' = \lambda$.

Proof. To simplify presentation, we shall prove only the case when $n = d = 1$, as the higher dimensional case can be treated in the same way without difficulty. Without loss of generality, assume $\lambda \geq \lambda'$. Set $(\hat{Y}, \hat{Z}, \hat{\lambda}) = (Y^x - Y'^x, Z^x - Z'^x, \lambda - \lambda')$. Then we have for each T ,

$$\hat{Y}_t + K_t'^x = \hat{Y}_T + K_T'^x + \int_t^T [\hat{f}_s + \gamma^1 \hat{\lambda}] ds + \int_t^T [\hat{g}_s + \gamma^2 \hat{\lambda}] d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^x - K_t^x),$$

where $\hat{f}_s = f(X_s^x, Z_s^x) - f(X_s^x, Z_s'^x) = b_s^\epsilon \hat{Z}_s + m_s^\epsilon$, $\hat{g}_s = g(X_s^x, Z_s^x) - g(X_s^x, Z_s'^x) = d_s^\epsilon \hat{Z}_s + n_s^\epsilon$, $|m_s^\epsilon| \leq 2\alpha_2\epsilon$ and $|n_s^\epsilon| \leq 2\alpha_2\epsilon$. By a similar analysis as in Theorem 3.3, we obtain

$$\begin{aligned} \hat{Y}_0 &\leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon \hat{Y}_T + \int_0^T \gamma^1 \hat{\lambda} X_s^\epsilon ds + \int_0^T \gamma^2 \hat{\lambda} X_s^\epsilon d\langle B \rangle_s] + \hat{\mathbb{E}}^{\tilde{G}}[\int_0^T m_s^\epsilon X_s^\epsilon ds + \int_0^T n_s^\epsilon X_s^\epsilon d\langle B \rangle_s] \\ &\leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon \hat{Y}_T] + \hat{\mathbb{E}}^{\tilde{G}}[\int_0^T (\gamma^1 \hat{\lambda} + 2G(\gamma^2 \hat{\lambda})) X_s^\epsilon ds] + 2(1 + \bar{\sigma}^2)\alpha_2 T\epsilon, \end{aligned}$$

where $\{X_t^\epsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\epsilon = 1 + \int_0^t d_s^\epsilon X_s^\epsilon dB_s + \int_0^t b_s^\epsilon X_s^\epsilon d\tilde{B}_s.$$

By the G -Itô's formula, we derive that $\hat{\mathbb{E}}^{\tilde{G}}[\int_0^T (\gamma^1 \hat{\lambda} + 2G(\gamma^2 \hat{\lambda})) X_s^\epsilon ds] = \hat{\mathbb{E}}^{\tilde{G}}[(\gamma^1 + 2G(\gamma^2)) \hat{\lambda} X_T^\epsilon T] = (\gamma^1 + 2G(\gamma^2)) \hat{\lambda} T$. Consequently,

$$\hat{Y}_0 \leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon \hat{Y}_T] + (\gamma^1 + 2G(\gamma^2)) \hat{\lambda} T + 2(1 + \bar{\sigma}^2)\alpha_2 T\epsilon.$$

Recalling Lemma 4.1, there exists some constant C such that

$$\hat{\mathbb{E}}^{\tilde{G}}[|X_T^\epsilon \hat{Y}_T|] \leq C(1 + |x|).$$

Thus letting $\epsilon \rightarrow 0$, we can find some constant C depending on G and c_x, \bar{C} such that for each T ,

$$\lambda - \lambda' \leq \frac{C}{T}(1 + |x|).$$

So if we let $T \rightarrow \infty$ in the above inequality, we conclude that $\lambda \leq \lambda'$, which concludes the result. ■

6 Applications

6.1 Large time behaviour of solutions to fully nonlinear PDEs

In this section, we shall study the large time behaviour of solutions of fully nonlinear PDEs by the theory of G -EBSDE. Let us introduce the following G -EBSDE:

$$Y_s^x = Y_T^x + \int_s^T [f(X_r^x, Z_r^x) - \lambda] dr + \int_s^T g_{ij}(X_r^x, Z_r^x) d\langle B^i, B^j \rangle_r - \int_s^T Z_r^x dB_r - (K_T^x - K_s^x), \quad (21)$$

and the fully nonlinear ergodic PDE:

$$G(H(D_x^2 v, D_x v, x)) + \langle b(x), D_x v \rangle + f(x, \langle \sigma_1(x), D_x v \rangle, \dots, \langle \sigma_d(x), D_x v \rangle) = \lambda, \quad (22)$$

where

$$H_{ij}(D_x^2 v, D_x v, x) = \langle D_x^2 v \sigma_i(x), \sigma_j(x) \rangle + 2\langle D_x v, h_{ij}(x) \rangle + 2g_{ij}(x, \langle \sigma_1(x), D_x v \rangle, \dots, \langle \sigma_d(x), D_x v \rangle).$$

From the section 5, we obtain that the G -EBSDE (21) and the fully nonlinear ergodic PDE (22) both have solutions. Moreover, the constant λ in the ergodic equation (22) is unique.

For each Lipschitz function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$, consider the following fully nonlinear parabolic PDE:

$$\begin{cases} \partial_t u(t, x) - G(H(D_x^2 u, D_x u, x)) - \langle b(x), D_x u \rangle - f(x, \langle \sigma_1(x), D_x u \rangle, \dots, \langle \sigma_d(x), D_x u \rangle) = 0, \\ u(0, x) = \varphi(x). \end{cases} \quad (23)$$

Denote $u^T(t, x) := u(T - t, x)$ for each $T > 0$. Then $u^T(t, x)$ is the unique viscosity solution of PDE:

$$\begin{cases} \partial_t u^T(t, x) + G(H(D_x^2 u^T, D_x u^T, x)) + \langle b(x), D_x u^T \rangle + f(x, \langle \sigma_1(x), D_x u^T \rangle, \dots, \langle \sigma_d(x), D_x u^T \rangle) = 0, \\ u^T(T, x) = \varphi(x). \end{cases}$$

Theorem 6.1 *Under assumptions (B1), (B2), (B4) and (B5), there exists a constant C such that, for each $T > 0$,*

$$|\frac{u(T, x)}{T} - \lambda| \leq \frac{C(1 + |x|)}{T}.$$

In particular,

$$\lim_{T \rightarrow \infty} \frac{u(T, x)}{T} = \lambda.$$

Proof. For convenience, assume $n = d = 1$. Recalling nonlinear Feynman-Kac formula in [13], we obtain for each $s \in [0, T]$,

$$u^T(s, X_s^x) = \varphi(X_T^x) + \int_s^T f(X_r^x, Z_r^{x,T}) dr + \int_s^T g(X_r^x, Z_r^{x,T}) d\langle B \rangle_r - \int_s^T Z_r^{x,T} dB_r - (K_T^{x,T} - K_s^{x,T}).$$

From equation (20), we conclude

$$\hat{Y}_t + K_t^x = \hat{Y}_T + K_T^x + \int_t^T [\hat{f}_s + \lambda] ds + \int_t^T \hat{g}_s d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^{x,T} - K_t^{x,T}),$$

where $(\hat{Y}, \hat{Z}) = (u^T(\cdot, X^\cdot) - Y^x, Z^{x,T} - Z^x)$, $\hat{f}_s = f(X_s^x, Z_s^{x,T}) - f(X_s^x, Z_s^x) = b_s^\epsilon \hat{Z}_s + m_s^\epsilon$ and $\hat{g}_s = g(X_s^x, Z_s^{x,T}) - g(X_s^x, Z_s^x) = d_s^\epsilon \hat{Z}_s + n_s^\epsilon$ for each $\epsilon > 0$. By a standard argument, we derive that, in the extended space,

$$\hat{Y}_0 \leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon \hat{Y}_T + \int_0^T \lambda X_s^\epsilon ds] + 2(1 + \bar{\sigma}^2)LT\epsilon = \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon \hat{Y}_T] + \lambda T + 2(1 + \bar{\sigma}^2)\alpha_2 T\epsilon,$$

where $\{X_t^\epsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\epsilon = 1 + \int_0^t d_s^\epsilon X_s^\epsilon dB_s + \int_0^t b_s^\epsilon X_s^\epsilon d\tilde{B}_s.$$

Denote by C_0 a constant that depends only on v and φ , which is allowed to change from line to line. Consequently, we have

$$u(T, x) - v(x) - \lambda T \leq \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon |\varphi(X_T^x) - v(X_T^x)|] + 2(1 + \bar{\sigma}^2)\alpha_2 T \epsilon \leq C_0 \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon (1 + |X_T^x|)] + 2(1 + \bar{\sigma}^2)\alpha_2 T \epsilon.$$

In a similar way, we can also get

$$v(x) + \lambda T - u(T, x) \leq C_0 \hat{\mathbb{E}}^{\tilde{G}}[X_T^\epsilon (1 + |X_T^x|)] + 2(1 + \bar{\sigma}^2)\alpha_2 T \epsilon.$$

Sending $\epsilon \rightarrow 0$ and recalling Lemma 4.1, there exists some constant C depending on G and M, C_0, L such that for each T ,

$$|u(T, x) - v(x) - \lambda T| \leq C(1 + |x|),$$

which gives the result. ■

Remark 6.2 Suppose $f(x, z)$ and $g(x, z)$ are independent of z . One can easily show that

$$\begin{aligned} v^\epsilon(x) &= \lim_{T \rightarrow \infty} \hat{\mathbb{E}}[\exp(-\epsilon T) Y_T^{x, \epsilon} + \int_0^T \exp(-\epsilon s) f(X_s^x) ds + \int_0^T \exp(-\epsilon s) g_{ij}(X_s^x) d\langle B^i, B^j \rangle_s] \\ &= \hat{\mathbb{E}}[\int_0^\infty \exp(-\epsilon s) f(X_s^x) ds + \int_0^\infty \exp(-\epsilon s) g_{ij}(X_s^x) d\langle B^i, B^j \rangle_s]. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{T} \hat{\mathbb{E}}[\int_0^T f(X_s^x) ds + \int_0^T g_{ij}(X_s^x) d\langle B^i, B^j \rangle_s] \\ &= \lim_{\epsilon \rightarrow 0} \epsilon \hat{\mathbb{E}}[\int_0^\infty \exp(-\epsilon s) f(X_s^x) ds + \int_0^\infty \exp(-\epsilon s) g_{ij}(X_s^x) d\langle B^i, B^j \rangle_s] \\ &= \lambda, \end{aligned}$$

which can be seen as Abelian-Tauberian Theorem for G -expectation.

6.2 Optimal ergodic control under model uncertainty

The objective of this section is to study optimal ergodic control problems under the model uncertainty. Let U be a closed subset of \mathbb{R}^n . We define a control $u_s \in M_G^2(0, \infty)$ as a U -valued process. Let $R : U \mapsto \mathbb{R}^d$ and $\kappa : \mathbb{R}^n \times U \mapsto \mathbb{R}$ be two bounded L -Lipschitz functions. Moreover, $|R(u)| \leq \alpha_2$. For each control $u_s \in M_G^2(0, \infty)$, we introduce the following Girsanov transformation under G -expectation framework, which is given in [13]. For each $T > 0$ and $\xi \in L_G^2(\Omega_T)$, consider the following G -BSDE:

$$Y_t = \xi + \int_t^T R(u_s) Z_s ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

Then $\tilde{\mathbb{E}}_t^u[\xi] := Y_t$ is a consistent sublinear expectation and $B_t^u := B_t - \int_0^t R(u_s) ds$ is a G -Brownian motion under $\tilde{\mathbb{E}}^u$.

Under the model uncertainty, the nonlinear ergodic cost corresponding to u and the starting point x is

$$J(x, u) = \limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\mathbb{E}}^u \left[\int_0^T \kappa(X_s^x, u_s) ds \right]. \quad (24)$$

Our purpose is to minimize costs J over all controls. Then define the Hamiltonian in the usual way

$$f(x, z) = \inf_u (\kappa(x, u) + R(u)z). \quad (25)$$

From section 5, the G -EBSDE (21) ($g = 0$) has a solution (Y^x, Z^x, K^x, λ) such that

$$|Y_s^x| \leq M |X_s^x|.$$

Theorem 6.3 *Suppose assumptions (B1), (B2), (B4) and (B5) hold. If for some $x \in \mathbb{R}^n$, $(Y, Z, K, \lambda') \in \mathfrak{S}_G^2(0, \infty) \times \mathbb{R}$ satisfies equation (21). Moreover, there exists a constant $c^x > 0$ such that*

$$|Y_s| \leq c^x (1 + |X_s^x|).$$

Then for any control $u \in M_G^2(0, \infty)$, we have $J(x, u) \geq \lambda' = \lambda$, and the equality holds if and only if for almost every t

$$f(X_t^x, Z_t) = \kappa(X_t^x, u_t) + R(u_t)Z_t.$$

Proof. It is obvious that $\lambda' = \lambda$. Since (Y, Z, K, λ) is a solution of the ergodic G -BSDE (21), we have

$$\begin{aligned} Y_s &= Y_T + \int_s^T [f(X_r^x, Z_r) - \lambda] dr - \int_s^T Z_r dB_r - (K_T - K_s) \\ &= Y_T + \int_s^T [f(X_r^x, Z_r) - \lambda] dr - \int_s^T Z_r dB_r^u - \int_s^T Z_r R(u_r) dr - (K_T - K_s), \end{aligned}$$

Consequently,

$$\lambda T + \tilde{\mathbb{E}}^u[K_T] = \tilde{\mathbb{E}}^u[Y_T - Y_0 + \int_0^T [f(X_r^x, Z_r) - Z_r R(u_r)] dr].$$

Note that $\tilde{\mathbb{E}}^u[K_T] = 0$, we obtain

$$\lambda \leq \frac{1}{T} \tilde{\mathbb{E}}^u[Y_T - Y_0 + \int_0^T \kappa(X_s^x, u_s) ds].$$

From Remark 5.3 in [13] and Lemma 4.1, we have $\tilde{\mathbb{E}}^u[|Y_T|] \leq C(1 + |x|)$. Consequently,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \tilde{\mathbb{E}}^u[|Y_T - Y_0|] = 0.$$

Thus, we obtain that

$$J(x, u) = \limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\mathbb{E}}^u \left[\int_0^T \kappa(X_s^x, u_s) ds \right] \geq \lambda.$$

In particular, if $f(X_t^x, Z_t) = \kappa(X_t^x, u_t) + R(u_t)Z_t$, we derive that

$$\lambda = \limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\mathbb{E}}^u[Y_T - Y_0 + \int_0^T \kappa(X_s^x, u_s) ds] = J(x, u),$$

which completes the proof. ■

Remark 6.4 From the above proof, if \limsup is changed into \liminf in the equation (24), then the same results hold. Moreover, the optimal value is given by λ in both cases.

Appendix

A.1 The proof of Lemma 4.3

In order to prove Lemma 4.3, we consider the following type of G -FBSDEs with infinite horizon: for each $t \geq 0$ and $\xi \in L_G^4(\Omega_t)$,

$$\begin{cases} X_s^{t,\xi} = \xi + \int_t^s b(X_r^{t,\xi})dr + \int_t^s h_{ij}(X_r^{t,\xi})d\langle B^i, B^j \rangle_r + \int_t^s \sigma(X_r^{t,\xi})dB_r, \\ Y_s^{t,\xi} = Y_T^{t,\xi} + \int_s^T f(X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})dr + \int_s^T g_{ij}(X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})d\langle B^i, B^j \rangle_r \\ \quad - \int_s^T Z_r^{t,\xi}dB_r - (K_T^{t,\xi} - K_s^{t,\xi}). \end{cases}$$

Using the same method as in Lemma 4.2, we have the following.

Lemma A.1 *Under assumptions (B1)-(B5), there exists a constant M depending only on $L, \alpha_1, \alpha_2, \eta$ and G such that*

$$|Y_t^{t,\xi} - Y_t^{t,\xi'}| \leq M|\xi - \xi'|.$$

Set

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

Lemma A.2 *$u(t, x)$ is a deterministic function of (t, x) . Moreover, $u(t, x) = u(x)$ for each $t \geq 0$.*

Proof. Denote by $(Y^{n,x}, Z^{n,x}, K^{n,x})$ the unique solution of the following G -BSDE in $[0, n]$:

$$Y_s^{n,x} = \int_s^n f(X_r^x, Y_r^{n,x}, Z_r^{n,x})dr + \int_s^n g_{ij}(X_r^x, Y_r^{n,x}, Z_r^{n,x})d\langle B^i, B^j \rangle_r - \int_s^n Z_r^{n,x}dB_r - (K_n^{n,x} - K_s^{n,x}),$$

and $(Y^{n,t,x}, Z^{n,t,x}, K^{n,t,x})$ the unique solution of the following G -BSDE in $[t, n+t]$:

$$\begin{aligned} Y_s^{n,t,x} &= \int_s^{n+t} f(X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})dr + \int_s^{n+t} g_{ij}(X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})d\langle B^i, B^j \rangle_r - \int_s^{n+t} Z_r^{n,t,x}dB_r \\ &\quad - (K_{n+t}^{n,t,x} - K_s^{n,t,x}). \end{aligned}$$

By the proof of Theorem 3.3, we get $\lim_{n \rightarrow \infty} Y_0^{n,x} = u(x)$ and $\lim_{n \rightarrow \infty} Y_t^{n,t,x} = u(t, x)$. Since $(B_{t+s} - B_t)_{s \geq 0}$ is also a G -Brownian motion, we have $Y_0^{n,x} = Y_t^{n,t,x}$. Thus $u(x) = u(t, x)$, which completes the proof. ■

Lemma A.3 *For each $\xi \in L_G^4(\Omega_t)$, we have*

$$u(\xi) = Y_t^{t,\xi}.$$

Proof. By Lemma A.1, we only need to prove Lemma A.3 for bounded $\xi \in L_G^4(\Omega_t)$. Thus for each $\varepsilon > 0$, we can choose a simple function $\eta^\varepsilon = \sum_{i=1}^N x_i \mathbf{1}_{A_i}$, where $(A_i)_{i=1}^N$ is a $\mathcal{B}(\Omega_t)$ -partition and $x_i \in \mathbb{R}^n$, such that $|\eta^\varepsilon - \xi| \leq \varepsilon$. It follows from Lemma A.1 that

$$|Y_t^{t,\xi} - u(\eta^\varepsilon)| = |Y_t^{t,\xi} - \sum_{i=1}^N u(x_i) \mathbf{1}_{A_i}| = |Y_t^{t,\xi} - \sum_{i=1}^N Y_t^{t,x_i} \mathbf{1}_{A_i}| = \sum_{i=1}^N |Y_t^{t,\xi} - Y_t^{t,x_i}| \mathbf{1}_{A_i} \leq M\varepsilon.$$

Noting that $|u(\xi) - u(\eta^\varepsilon)| \leq M\varepsilon$, we get $|Y_t^{t,\xi} - u(\xi)| \leq 2M\varepsilon$. Since ε can be arbitrarily small, we obtain $Y_t^{t,\xi} = u(\xi)$. ■

The proof of Lemma 4.3. It is easy to check that $X_s^{t,X_t^x} = X_s^x$ for $s \geq t$. Then by the uniqueness of G -BSDE (13), we obtain $Y_t^{t,X_t^x} = Y_t^x$, which yields the desired result by applying Lemma A.3. ■

A.2 Uniqueness of viscosity solution to fully nonlinear elliptic PDEs

Theorem A.4 *Under assumptions (B1)-(B5), if $\tilde{u}(x)$ is a bounded continuous viscosity solution to equation (15), then*

$$u = \tilde{u}.$$

In order to prove Theorem A.4, we need the following lemmas.

Lemma A.5 *For each bounded and continuous function $\phi \in C_b(\mathbb{R}^n)$, $\hat{\mathbb{E}}[\phi(X_t^x)]$ is a continuous function of (t, x) .*

Proof. Assume ϕ is bounded by $M > 0$. For each given $N > 0$ and $T > 0$, for any $t, t' < T$, $x, x' \in \mathbb{R}^n$, we have

$$\begin{aligned} |\hat{\mathbb{E}}[\phi(X_t^x)] - \hat{\mathbb{E}}[\phi(X_{t'}^{x'})]| &\leq \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})|] \\ &\leq \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})| \mathbf{1}_{\{|X_t^x| \leq N\} \cap \{|X_{t'}^{x'}| \leq N\}}] \\ &\quad + \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})| (\mathbf{1}_{\{|X_t^x| \geq N\}} + \mathbf{1}_{\{|X_{t'}^{x'}| \geq N\}})] \\ &\leq \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})| \mathbf{1}_{\{|X_t^x| \leq N\} \cap \{|X_{t'}^{x'}| \leq N\}}] + \frac{2M}{N} (\hat{\mathbb{E}}[|X_{t'}^{x'}|] + |X_t^x|). \end{aligned}$$

Note ϕ is uniformly continuous on $\{x : |x| \leq N\}$. Then for each given $\epsilon > 0$, there is $\rho > 0$ such that

$$|\phi(z) - \phi(z')| \leq \frac{\epsilon}{2} \text{ whenever } |z - z'| < \rho \text{ and } |z|, |z'| \leq N.$$

From Proposition 4.1 in [13], we obtain

$$\hat{\mathbb{E}}[|X_t^x - X_{t'}^{x'}|] \leq C_T(|t - t'|^{\frac{1}{2}} + |x - x'|),$$

where C_T depends on L, α_1, G, n and T . Then, by Chebyshev's inequality, there is $\delta > 0$ such that

$$c(|X_t^x - X_{t'}^{x'}| \geq \rho) < \frac{\epsilon}{4M}$$

whenever $|x - x'| \leq \delta$ and $|t - t'| \leq \delta$. Consequently,

$$\begin{aligned} |\hat{\mathbb{E}}[\phi(X_t^x)] - \hat{\mathbb{E}}[\phi(X_{t'}^{x'})]| &\leq \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})| \mathbf{1}_{\{|X_t^x - X_{t'}^{x'}| < \rho\} \cap \{|X_t^x| \leq N\} \cap \{|X_{t'}^{x'}| \leq N\}}] \\ &\quad + \hat{\mathbb{E}}[|\phi(X_t^x) - \phi(X_{t'}^{x'})| \mathbf{1}_{\{|X_t^x - X_{t'}^{x'}| \geq \rho\}}] + \frac{2M}{N} (\hat{\mathbb{E}}[|X_{t'}^{x'}|] + |X_t^x|) \\ &\leq \epsilon + \frac{2M}{N} (\hat{\mathbb{E}}[|X_{t'}^{x'}|] + |X_t^x|) \end{aligned}$$

whenever $|x - x'| \leq \delta$ and $|t - t'| \leq \delta$. Thus we get

$$\limsup_{(t', x') \rightarrow (t, x)} |\hat{\mathbb{E}}[\phi(X_t^x)] - \hat{\mathbb{E}}[\phi(X_{t'}^{x'})]| \leq \epsilon + \frac{2M}{N} (\hat{\mathbb{E}}[|X_{t'}^{x'}|] + |X_t^x|).$$

The proof is complete by letting $\epsilon \downarrow 0$ and then $N \rightarrow \infty$. ■

Now we consider the following type of G -BSDEs on $[0, T]$ with $T > 0$: for each $t \in [0, T]$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} Y_s^{t, T, x} = & \phi(X_T^{t, x}) + \int_s^T f(X_r^{t, x}, Y_r^{t, T, x}, Z_r^{t, T, x}) dr + \int_s^T g_{ij}(X_r^{t, x}, Y_r^{t, T, x}, Z_r^{t, T, x}) d\langle B^i, B^j \rangle_r \\ & - \int_s^T Z_r^{t, T, x} dB_r - (K_T^{t, T, x} - K_s^{t, T, x}), \end{aligned} \tag{26}$$

where ϕ is a continuous function bounded by $M > 0$. In particular, denote $(Y^{T,x}, Z^{T,x}, K^{T,x}) = (Y^{0,T,x}, Z^{0,T,x}, K^{0,T,x})$. Then we denote $\bar{u}(t, x) = Y_t^{t,T,x}$. Moreover, there exists a sequence Lipschitz functions $\{\phi^m\}_{m=1}^\infty$ bounded by M such that

$$|\phi(x) - \phi^m(x)| \leq \frac{1}{m} \mathbf{1}_{\{|x| \leq m\}} + 2M \mathbf{1}_{\{|x| > m\}}.$$

Let $(Y^{t,T,m,x}, Z^{t,T,m,x}, K^{t,T,m,x})$ be the unique $\mathfrak{S}_G^2(0, T)$ -solution of G -FBSDEs (26) with terminal condition $Y_T^{t,T,m,x} = \phi^m(X_T^{t,x})$ and denote $\bar{u}^m(t, x) = Y_t^{t,T,m,x}$.

Lemma A.6 ([13]) *Under assumptions (B1) and (B2), $\bar{u}^m(t, x)$ is the unique viscosity solution of the following fully nonlinear PDE with terminal condition $\bar{u}^m(T, x) = \phi^m(x)$:*

$$\begin{cases} \partial_t u + G(H(D_x^2 u, D_x u, u, x)) + \langle b(x), D_x u \rangle + f(x, u, \langle \sigma_1(x), D_x u \rangle, \dots, \langle \sigma_d(x), D_x u \rangle) = 0, \\ u(T, x) = \phi(x). \end{cases} \quad (27)$$

Moreover, $\bar{u}^m(t, X_t^x) = Y_t^{t,T,m,x}$ for each $t \in [0, T]$.

Lemma A.7 *Assume (B1) and (B2) hold. Then we have*

(1) *There exists a constant C depending on M, T, G, L, α and α_2 such that*

$$\|Y^{t,T,m,x}\|_{S_G^2(t,T)} + \|Z^{t,T,m,x}\|_{M_G^2(t,T)} + \|Y^{t,T,x}\|_{S_G^2(t,T)} + \|Z^{t,T,x}\|_{M_G^2(t,T)} \leq C, \quad \forall x \in \mathbb{R}^n, m \geq 1;$$

(2) $\lim_{m \rightarrow \infty} \hat{\mathbb{E}}[\sup_{s \in [t, T]} |Y_s^{t,T,m,x} - Y_s^{t,T,x}|^2] = 0$;

(3) $\bar{u}(t, x)$ is a bounded and continuous function;

(4) $\lim_{m \rightarrow \infty} \bar{u}^m(t_m, x_m) = \bar{u}(t, x)$ for each given $(t_m, x_m) \in [0, T] \times \mathbb{R}^n$ with $(t_m, x_m) \rightarrow (t, x)$.

Proof. Note that ϕ^m and $f(x, 0, 0), g_{ij}(x, 0, 0)$ is uniformly bounded. Applying Proposition 3.5 and Corollary 5.2 in [12], we obtain (1). By Theorem 2.4 and Theorem 3.3 in [28], we can find a constant \tilde{C} depending on M, T, G, L, α and α_2 (may vary from line to line), such that,

$$\begin{aligned} \lim_{m \rightarrow \infty} \hat{\mathbb{E}}[\sup_{s \in [t, T]} |Y_s^{t,T,m,x} - Y_s^{t,T,x}|^2] &\leq \lim_{m \rightarrow \infty} \tilde{C}((\hat{\mathbb{E}}[|\phi(X_T^{t,x}) - \phi^m(X_T^{t,x})|^3])^{\frac{2}{3}} + \hat{\mathbb{E}}[|\phi(X_T^{t,x}) - \phi^m(X_T^{t,x})|^3]) \\ &\leq \lim_{m \rightarrow \infty} \tilde{C}(\frac{1}{m^2} + \frac{\hat{\mathbb{E}}[|X_T^{t,x}|^3] + (\hat{\mathbb{E}}[|X_T^{t,x}|^3])^{\frac{2}{3}}}{m^2}) = 0. \end{aligned} \quad (28)$$

In particular, $\lim_{m \rightarrow \infty} \bar{u}^m(t, x) = \bar{u}(t, x)$.

Now we prove $\lim_{m \rightarrow \infty} \bar{u}(t_m, x_m) = \bar{u}(t, x)$ for each given $(t_m, x_m) \in [0, T] \times \mathbb{R}^n$ with $(t_m, x_m) \rightarrow (t, x)$. Without loss of generality, we assume $t_m \leq t$ and $g_{ij} = 0$. Using the method as in (2) and Lemma A.5, we can obtain

$$\lim_{m \rightarrow \infty} \hat{\mathbb{E}}[\sup_{s \in [t, T]} |Y_s^{t_m,T,x_m} - Y_s^{t,T,x}|^2 + \int_t^T |Z_s^{t_m,T,x_m} - Z_s^{t,T,x}|^2 ds] = 0. \quad (29)$$

By equation (26), we have

$$\bar{u}(t, x) + (K_T^{t,T,x} - K_t^{t,T,x}) = \phi(X_T^{t,x}) + \int_t^T f(X_r^{t,x}, Y_r^{t,T,x}, Z_r^{t,T,x}) dr - \int_t^T Z_r^{t,T,x} dB_r.$$

Taking expectation on both sides yields that

$$\bar{u}(t, x) = \hat{\mathbb{E}}[\phi(X_T^{t,x}) + \int_t^T f(X_r^{t,x}, Y_r^{t,T,x}, Z_r^{t,T,x}) dr].$$

Consequently,

$$\begin{aligned} |\bar{u}(t, x) - \bar{u}(t_m, x_m)| &\leq \hat{\mathbb{E}}[|\phi(X_T^{t,x}) - \phi(X_T^{t_m, x_m})| + \int_{t_m}^t |f(X_r^{t_m, x_m}, Y_r^{t_m, T, x_m}, Z_r^{t_m, T, x_m})| dr \\ &\quad + \int_t^T |f(X_r^{t,x}, Y_r^{t,T,x}, Z_r^{t,T,x}) - f(X_r^{t_m, x_m}, Y_r^{t_m, T, x_m}, Z_r^{t_m, T, x_m})| dr] \\ &\leq \hat{\mathbb{E}}[(t - t_m)^{\frac{1}{2}} (\int_{t_m}^t 3(|f(X_r^{t_m, x_m}, 0, 0)|^2 + |LY_r^{t_m, T, x_m}|^2 + |\alpha_2 Z_r^{t_m, T, x_m}|^2) dr)^{\frac{1}{2}} \\ &\quad + \int_t^T (L|X_r^{t,x} - X_r^{t_m, x_m}| + L|Y_r^{t,T,x} - Y_r^{t_m, T, x_m}| + \alpha_2 |Z_r^{t,T,x} - Z_r^{t_m, T, x_m}|) dr \\ &\quad + |\phi(X_T^{t,x}) - \phi(X_T^{t_m, x_m})|]. \end{aligned}$$

By Lemma A.5, (1) and equation (29), we derive that

$$\lim_{m \rightarrow \infty} |\bar{u}(t, x) - \bar{u}(t_m, x_m)| = 0,$$

and u is a bounded continuous function.

From (3), we get that

$$\begin{aligned} \lim_{m \rightarrow \infty} |\bar{u}^m(t_m, x_m) - \bar{u}(t, x)| &\leq \lim_{m \rightarrow \infty} |\bar{u}^m(t_m, x_m) - \bar{u}(t_m, x_m)| + \lim_{m \rightarrow \infty} |\bar{u}(t_m, x_m) - \bar{u}(t, x)| \\ &= \lim_{m \rightarrow \infty} |\bar{u}^m(t_m, x_m) - \bar{u}(t_m, x_m)|. \end{aligned}$$

By equation (28), we obtain

$$\lim_{m \rightarrow \infty} |\bar{u}^m(t_m, x_m) - \bar{u}(t, x)| \leq \lim_{m \rightarrow \infty} \tilde{C} \left(\frac{1}{m} + \frac{\hat{\mathbb{E}}[|X_T^{t_m, x_m}|^3]^{\frac{1}{2}} + \hat{\mathbb{E}}[|X_T^{t_m, x_m}|^3]^{\frac{1}{3}}}{m} \right) = 0.$$

The proof is complete. ■

By Lemmas A.6, A.7, Theorem 6.1 in [5] and Proposition 4.3 in [7], we have the following result, which is the nonlinear Feynman-Kac formula for parabolic PDE.

Lemma A.8 *Under assumptions (B1) and (B2), $\bar{u}(t, x)$ is the unique viscosity solution of the fully nonlinear PDE (27) with terminal condition $\bar{u}(T, x) = \phi(x)$. In particular, $\bar{u}(t, X_t^x) = Y_t^{T,x}$*

Now we give the proof of Theorem A.4.

The proof of Theorem A.4. For each $T > 0$, by Theorem 6.1 in [5], we obtain \tilde{u} is the unique viscosity solution of the fully nonlinear PDE (27) with terminal condition $\phi(x) = \tilde{u}(x)$. Then it follows Lemma A.8, $\tilde{u}(X_t^x) = Y_t^{T,x}$ for each $t \in [0, T]$, where

$$\begin{aligned} Y_s^{T,x} &= \tilde{u}(X_T^x) + \int_s^T f(X_r^x, Y_r^{T,x}, Z_r^{T,x}) dr + \int_s^T g_{ij}(X_r^x, Y_r^{T,x}, Z_r^{T,x}) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^T Z_r^{T,x} dB_r - (K_T^{T,x} - K_s^{T,x}). \end{aligned} \tag{30}$$

By the uniqueness of solution to G -BSDE in finite horizon, it is obvious $(Z_r^{x,T}, K_r^{x,T}) = (Z_r^{x,S}, K_r^{x,S})$ for $S > T$. Set $(Z_t^x, K_t^x) = (Z_t^{x,T}, K_t^{x,T})$ for some $T \geq t$. Then $(\tilde{u}(X_t^x), Z_t^x, K_t^x)_{t \geq 0}$ satisfies equation (13). Applying Theorem 3.3, we obtain $\tilde{u}(X_t^x) = u(X_t^x)$. In particular, $\tilde{u}(x) = u(x)$, which is the desired result. ■

Remark A.9 In this section, we introduce a new method to prove the uniqueness of the viscosity solutions to elliptic PDEs in \mathbb{R}^n , which non-trivially generalize the ones of [17] for fully nonlinear case. In particular, this method can be applied to deal with more general elliptic PDEs, for example, u is of polynomial growth.

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